2nd TED4LAT
Workshop and doctoral school
Paris, 21 – 25 October 2024

Institut des Systèmes Complexes Paris Île-de-France

Politecnico di Torino 1859

Introduction to Quantum Finance

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Introduction

- **Random walks** represent a valuable mathematical tool for describing how behavior at the microscopic level can influence macroscopic phenomena in a multiscale framework.
- **Quantum walks** represent quantum counterpart of random walks and serve as a valuable tool for illustrating the advantages of quantum approaches.
- **Quantum Finance** is a newly developed interdisciplinary subject that directly take advantages on quantum properties.
- **Aims**: Providing a clear, engaging, and straightforward explanation of quantum computing performances exploiting the transition from the classical to the quantum model of random walks. This brief review includes key definitions and properties, followed by a demonstration of a quantum model for financial applications.

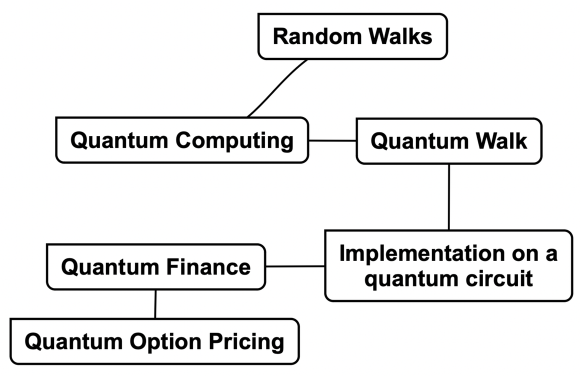


Figure 1. The flow chart outlines the progression of the main themes connecting key concepts.

Random Walks

- A simple random walk in one dimension can be defined as:

$$S_n = x + \sum_{j=1}^{n} X_j$$

An important result is the **Local Central Limit Theorem**, which states that, as the number of steps n increases, the distribution of the walk converges to a normal distribution:

$$P_n(X) \approx \frac{1}{\sqrt{2\pi n\sigma^2}} e^{-\frac{(x-n\langle x\rangle)^2}{2n\sigma^2}}$$

This theorem indicates that, for large n, the behavior of the random walk can be approximated by a Gaussian distribution, where σ^2 is the variance of the step distribution that is $\sigma^2 = x$.
In addition to random walks, we can consider continuous-time stochastic processes like **Brownian Motion** ($B(t)$). $B(t)$ can be described as the limit of a random walk. Its key properties are:
  - $B(0) = 0$
  - $B(t)$ has independent increments (*Markovian Process*).
  - $B(t)$ is a *Martingale* (expectation value of the final value is the current value)
  - $B(t) \sim N(0, t)$
- A related process is the **Geometric Brownian Motion (GBM)**. It can be defined as:

$$dX = \mu dt + \sigma dB_t$$

where μ is the drift and σ is the volatility. It is widely used in finance for modeling stock prices. GBM is defined in logarithmic scale so it ensures that the modeled quantity remains non-negative:

$$Y \sim N(\mu, \sigma^2),\ X = e^Y \text{ and } X = GBM$$

Quantum Computing

- Before introducing the quantum analogue of random walks, it is essential to understand the basic concepts of **quantum computing**. The key difference is the use of **qubits** rather than classical bits:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

where α and β are complex numbers, and $|\alpha|^2 + |\beta|^2 = 1$. This superposition allows quantum computers to process many possibilities simultaneously, exploiting the phenomenon of **quantum parallelism**.
- The evolution of qubits is governed by **quantum gates**, which are analogous to classical logic gates but operate according to unitary transformations. Main ones are:

Hadamard Gate
$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

Table 1. Equal superposition

Rotations about X-Axis
$R_x(\theta) = \cos\left(\frac{\theta}{2}\right) I - i\sin\left(\frac{\theta}{2}\right) X$
Pauli X-Gate $= X = R_x(\pi) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Table 2. Bit Flip

Rotations about Y-Axis
$R_y(\theta) = \cos\left(\frac{\theta}{2}\right) I - i\sin\left(\frac{\theta}{2}\right) Y$
Pauli Y-Gate $= Y = R_y(\pi) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

Table 3. Bit Flip + Phase Flip

Rotations about Z-Axis
$R_z(\theta) = \cos\left(\frac{\theta}{2}\right) I - i\sin\left(\frac{\theta}{2}\right) Z$
Pauli Z-Gate $= Z = R_z(\pi) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Table 4. Phase Flip

Toffoli Gate
CNOT $= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

Table 5. Toffoli Gate: Controlled-NOT gate

Once the mathematical formulations of an algorithm are established, these elementary gates are employed in quantum circuits to translate theoretical models into practical solutions.

Quantum Walk

- Quantum walks (QW) represent the quantum analogue of classical random walks, introduced to explore structured - search problems. QW operator is:

$$U^n = [T(R \otimes I)]^n$$

  - **Quantum Coin R**: it is spanned by $|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
  - **Translation Operator**: $T = |\uparrow\rangle\langle\uparrow| \otimes \sum_j |x_{j+1}\rangle\langle x_j| + |\downarrow\rangle\langle\downarrow| \otimes \sum_j |x_{j-1}\rangle\langle x_j|$
- QWs have practical applications in areas such as graph traversal, where they improve hitting, commute, and mixing times compared to classical algorithms.

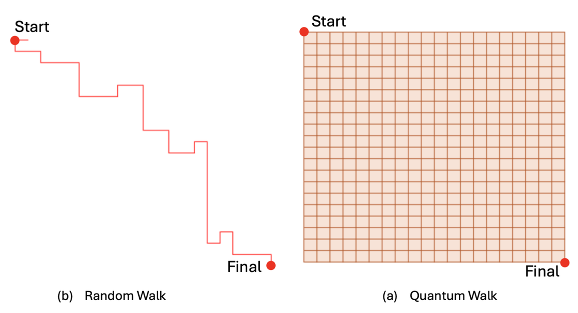


Figure 2. There are several different paths to reach the same position: the classical approach is determined step-by-step; quantum walk creates a superposition of all possible states and each position occurs with a certain probability value.

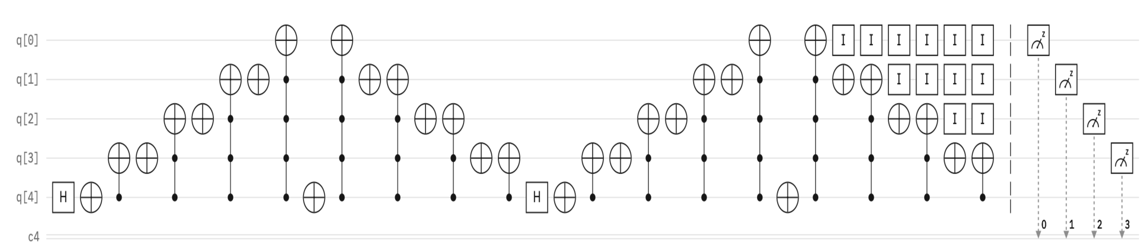
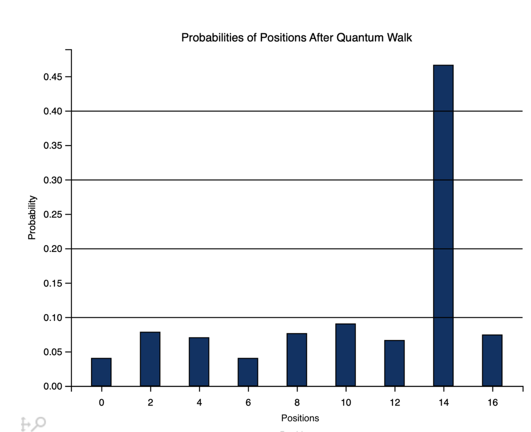
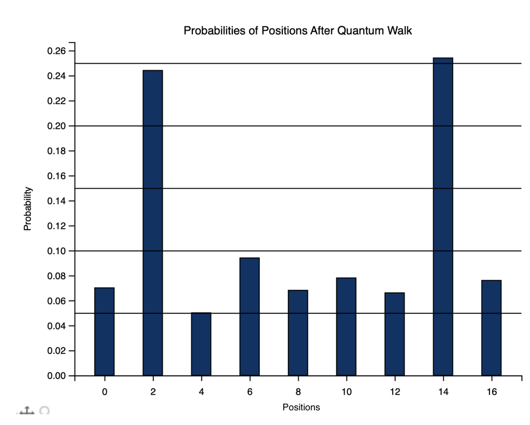


Figure 3. Increment and Decrement steps on IBM Composer: The qubit q[4] serves as the control qubit, while the remaining qubits (q[0], q[1], q[2], and q[3]) are the target qubits, representing the walk. By applying a Hadamard gate to q[4], we immediately create a superposition of states. Notably, no explicit initial state is provided for the qubits. During the increment phase (executed through a cascade of Toffoli gates from q[4] to q[0]), it is essential to apply a NOT gate to q[4] afterward. This resets the superposition on the control qubit, allowing us to proceed with the decrement phase (decreasing from q[0] to q[3]). The walk is iterated twice, followed by measurements on q[0], q[1], q[2], and q[3]. We avoid measuring q[4] since doing so would collapse the superposition and is unnecessary for this operation. (The I gates are identity gates used for clarity in visualization).



(a) Initial state: $|\Psi_0\rangle = |\uparrow\rangle \otimes |0\rangle$



(b) Initial state symmetric: $|\Psi_{sym}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + i|\downarrow\rangle)$

Figure 4. These distributions represent two distinct cases, highlighting the key differences compared to their classical counterparts. In Figure 3b, while the mean remains centered, similar to a normal distribution, the variance exhibits significantly different behavior. Classical probability distribution has '*slower*' variance which indicates that the quantum distribution has data points more spread out around the mean, signifying greater variability or uncertainty in the quantum model: $Var(t)_{QW} \simeq 0,54t^2$.

QWs offer a detailed way to understand how quantum computers perform computations. They require a specific graph structure for the walker to move through. This becomes an unnecessary constraint, especially in finance, as unstructured search algorithms such as Quantum Amplitude Estimation (QAE) can be more efficient and versatile.

Stochastic Processes Accelerated by Quantum Walk-based Search

- QWs are a useful tool for generalizing key properties of quantum computing but there exist simpler solutions that can be used when dealing with unstructured problems i.e. QAE. Let introduce a quantum solution for an important financial instrument: **option**.
- The **quantum option pricing** model utilizes quantum computing to accelerate the pricing of options. We choose to consider a *European Call Option*. When beginning this process, we choose the Black-Scholes method, which assumes a lognormal distribution of possible future asset prices. A new challenge now arises: preparing a quantum state using qubits, where each qubit represents an asset, and its value and associated probability are determined by the lognormal distribution previously given. This presented a difficult research challenge for which we still need to find a more efficient solution. However, a promising approach can facilitate the **loading** of generic probability distributions, implicitly provided by data samples, into quantum states (see figure 5a):

$$|0\rangle_n \rightarrow |\psi\rangle_n = \sum_{i=0}^{N-1} \sqrt{p_i}\,|i\rangle_n$$

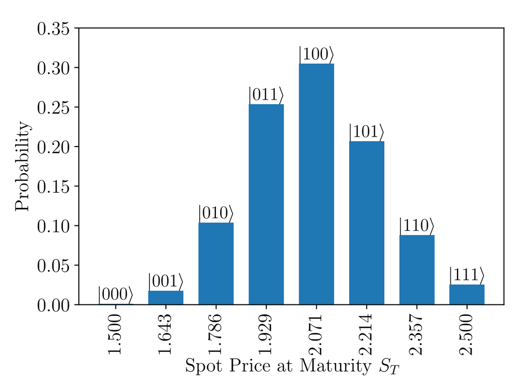
- We require a **Strike Price Comparator** that can decide which assets will contribute to the value of option. The resulting state is the following:

$$|\psi\rangle_n|0\rangle \rightarrow |\phi_1\rangle = \sum_{i<K} \sqrt{p_i}\,|i\rangle_n|0\rangle + \sum_{i\leq K} \sqrt{p_i}\,|i\rangle_n|1\rangle$$

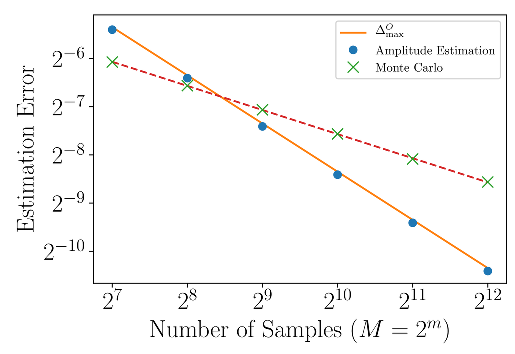
The qubit's amplitude represents the probability of the option being "in the money" (when the asset price exceeds the strike price).
- To implement the **payoff function** in a quantum circuit, we need to encode it into controlled rotations, where the angle of rotation of a state depends on the difference value. And we end up with:

$$|\phi_f\rangle = \sum_{i<K} \sqrt{p_i}\,|i\rangle_n(\cos(\theta_i)|0\rangle + \sin(\theta_i)|1\rangle)$$

This state encloses the most relevant probability amplitude about asset prices at maturity and so, QAE will be applied on this as initial state. We find that QAE converges very quickly to the result.



(a) Asset prices loaded and represented on qubit states.



(b) This is the estimation error that shows in loglogscale how it spread over the number of samples

Figure 5. a represents the probability distribution derived from the lognormal distribution and loaded by qGAN model. Figure 5b illustrates the comparison between the estimation errors of (QAE) and Monte Carlo methods. The estimation error refers to the difference between the value estimated by a method and the true or expected value of the option. In this case, the plot shows how the estimation error decreases as the number of samples increases for both methods. The key takeaway is that QAE demonstrates an $O(M^{-1})$ convergence, meaning that the estimation error decreases quadratically with the number of samples, while Monte Carlo simulations follow an $O(M^{-1/2})$ rate, indicating a slower reduction in error.

Final remarks

- We have explored how quantum computers perform calculations using random walks and their quantum analogs. We demonstrated how quantum algorithms can solve structured problems and that it generalizes the unstructured-search method of QAE, which offers a powerful solution for financial challenges. We concluded with a specific case—option pricing—by introducing how a European call option can be priced using quantum techniques.
  - ✓ The results demonstrated that quantum approach is quadratically faster than classical algorithms.
  - ✓ Quantum model offers a new probability distribution that encloses a higher variance and then values are more spread out around the mean.
  - ✓ In quantum option pricing, in its simplest terms (European call options), QAE reduces the number of queries needed for accurate payoff estimation.
  - ✗ While promising, hardware limitations in current quantum devices still present challenges. Ongoing advancements in quantum computing are essential for fully realizing its potential in financial applications.

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